International Journal of Theoretical Physics, Vol. 26, No. 7, 1987

Equations of Motion for Continuum Systems

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Received February 13, 1987

We construct equations of motion for an N-component continuum. The basic assumption is that the dynamical vector field is the sum of two terms: a "conservative" term, being a Hamiltonian vector field associated with the energy function of the system; and a "dissipative" term, being a gradient vector field associated with a family of functions. The resulting equations satisfy the usual conservation laws for continuum systems, and, moreover, reduce to the "standard" fluid equations when the continuum is a fluid.

1. INTRODUCTION

The notion of continuum comprises all the possible phases of matter: the gas, the liquid, and the solid phases. It is thus a relatively abstract concept. This is reflected in the theory of the continuum, being based on purely geometrical and thermodynamic considerations, without any particular reference to the properties of matter. The aim of this paper is not to discuss the foundations of continuum thermodynamics, but to give a derivation of the equations of motion. We assume that the dynamical vector field χ is the sum of two terms: a Hamiltonian vector field χ^H defined by the energy function and leaving the entropy invariant, and a "gradient" vector field χ^G describing the directed transformation of mechanical and other forms of energy into heat. The "dissipation function" r from which χ^G is constructed is given as the composite $r \circ \psi$, where ψ is a submersion containing the information about the dissipation mechanisms that are generally valid physical laws, while the function r contains the information about the dissipation coefficients.

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0020-7748/87/0700-0697$05.00/0

It is certainly a gain to be able to refer the complete information about the dynamics of a continuum system to a couple of functions. Our main motivation for this study has, however, been the construction of a dynamics for multicomponent systems. In fact, once we know the construction for a one-component system in a framework like this, then the generalization to the multicomponent case becomes trivial.

In the following we have chosen to expose the construction by writing it out for an N-component continuum. An interesting generalization would be to assume some or all of the components to be charged. This can easily be obtained using the results of Aaberge (1986). In fact, some of the systems for which one would like to give a more complete description are multicomponent systems with charged components, such as electric conductors, which consist of an ionic lattice and a conduction electron gas.

The state space of a thermodynamic system is a Banach manifold $\mathscr{B}(\mathscr{E})$ represented as a function space over space $X=\mathbb{R}^3$. The states thus are sections $\gamma: X \to \mathscr{E}$ of a fibered manifold $\pi: X \to \mathscr{E}$. Moreover, the extensive observables of the system are represented by functions $F: \mathscr{B}(\mathscr{E}) \to \mathbb{R};\ \gamma \to F(\gamma)=\int_\gamma f\,d^3x$, where $f: J_q(\mathscr{E}) \to \mathbb{R}$ is a function on the q-jet extension of $\mathscr{E}$, i.e., of γ and its derivatives up to order q. Now, given $\mathscr{E}$, $J_q\mathscr{E}$ is canonically given; J_q is a functor. Thus, morphisms of $\mathscr{E}$ are lifted by J_q to morphisms of $J_q(\mathscr{E})$ (Pommaret, 1978). There also exist functors $\mathscr{B}$ between the category of fibered manifolds and the category of Banach spaces (Palais, 1968). However, presently one does not know which one to choose, i.e., which topology to choose on $\mathscr{B}(\mathscr{E})$ in thermodynamics. Whatever this might be, a main part of thermodynamics can be formulated and a number of formal properties can be studied using the standard methods of differential calculus on $\mathscr{E}$ and $J_q(\mathscr{E})$. This is what we will do in the following. A brief exposition of these methods, and the notation we will use is found in an Appendix to Aaberge (1986).

2. DEFINITION OF THE SYSTEM

The local observables of the N-component continuum system are functions on the q-jet extension $J_q(\mathscr{E})$ of the fibered manifold (Aaberge, 1986; Pommaret, 1978; Palais, 1968)

$$\mathscr{E}_u=\{(x^i, s_n, \pi_{ni}, \alpha_{nij}) \in \mathbb{R}^3 \times \mathbb{R}^N \times \mathbb{R}^{3N} \times \mathbb{R}^{6N}\}$$

where α_{nij} is assumed to be a density-valued metric, and the fibration is defined by

$$\mathscr{E}_u \to X, \qquad (x^i, s_n, \pi_{ni}, \alpha_{nij}) \mapsto (x^i)$$

Accordingly, the local extensive observables of entropy-density s_n, momentum-density π_{ni}, the "moment" density α_{nij}, and the mass-density ρ_n of the nth component are represented by the functions $J_q(\mathscr{E}_u) \to \mathbb{R}$:

$$\hat{s}_n(x^i, s_n, \ldots, \alpha_{nij,i_1i_2\ldots i_q}) = s_n$$
$$\hat{\pi}_{ni}(x^i, s_n, \ldots, \alpha_{nij,i_1i_2\ldots i_q}) = \pi_{ni}$$
$$\hat{\alpha}_{nij}(x^i, s_n, \ldots, \alpha_{nij,i_1i_2\ldots i_q}) = \alpha_{nij}$$
$$\hat{\rho}_n(x^i, s_n, \ldots, \alpha_{nij,i_1i_2\ldots i_q}) = (\det \alpha_n)^{1/5}$$

In addition, we will need to consider the local observables of total entropy density s, total momentum density, total mass density ρ, and total angular momentum density l_i. These observables are represented by the functions

$$\hat{s}_n(x^i, s_n, \ldots) = \sum_{n=1}^{N} s_n$$
$$\hat{\pi}_i(x^i, s_n, \ldots) = \sum_{n=1}^{N} \pi_{ni}$$
$$\hat{\rho}(x^i, s_n, \ldots) = \sum_{n=1}^{N} (\det \alpha_n)^{1/5}$$
$$\hat{l}_i(x^i, s_n, \ldots) = \varepsilon_{ijk}\delta^{kl}x^j\pi_l$$

where ε and δ are the usual summation symbols.

According to the laws of thermodynamics, a thermodynamic system is defined by its energy function, i.e., in this setting by the energy density

$$\hat{u} : J_q(\mathscr{E}_u) \to \mathbb{R}$$

If the system is *isolated*, then $\hat{u}$ is homogeneous (i.e., $\partial_{x^i}\hat{u} = 0$) and invariant under rotations. In any case, we assume that $\hat{u}$ does not depend on the derivatives of s_n, $\nabla_{s_n}\hat{u} = \partial_{s_n}\hat{u}$.[3]

To a given set of extensive observables we have a corresponding set of local intensive observables. In the given representation called the energy representation, these are the temperature T_n, velocity v^i_n, and the "stress" σ^{ij}_n of the nth component. They are represented by the functions $J_q(\mathscr{E}_u) \to \mathbb{R}$:

$$\hat{T}_n(x^i, s_n, \ldots) = \partial_{s_n}\hat{u}(x^i, s_n, \ldots)$$
$$\hat{v}^i_n(x^i, s_n, \ldots) = \nabla_{\pi_{ni}}\hat{u}(x^i, s_n, \ldots)$$
$$\hat{\sigma}^{ij}_n(x^i, s_n, \ldots) = \nabla_{\alpha_{nij}}\hat{u}(x^i, s_n, \ldots)$$

Since $T_n = \partial_s\hat{u} > 0$, the map $\phi_{us} : J_q(\mathscr{E}_u) \to J_q(\mathscr{E}_s)$, where

$$\mathscr{E}_s = \{(x^i, u, \sigma_m, \pi_{ni}, \alpha_{nij}) | \cdots\}, \qquad \sigma_m = \sum_{n=1}^{N} s_n, \qquad m = 1, \ldots, N-1$$

[3] $\nabla_\pi = \partial_\pi - \nabla_i \partial_{\pi_{,i}} + \nabla_i \nabla_j \partial_{\pi_{,ij}} \cdots$.

is a diffeomorphism. On $\mathscr{E}_s$, u appears as a coordinate function and s as a potential function.

3. THE DYNAMICAL POSTULATE

The evolution of a thermodynamic system during a time interval $[t_0, t_1]$ is described by a curve c on the state space $\mathscr{B}(\mathscr{E})$. The curve c is a generalized solution of an ordinary differential equation on $\mathscr{B}(\mathscr{E})$, the equation of motion

$$\dot{c} = \chi \circ j_q(c)$$

where χ denotes the symbol for the dynamical vector field on $\mathscr{B}(\mathscr{E})$. The $\chi = \chi^{\alpha} \nabla_{y^{\alpha}}$ is represented as a differential operator on $J_q(\mathscr{E})$; its coefficients χ^{α} are functions on $J_q(\mathscr{E})$.

The only explicit assumption on the dynamics usually given is the following version of the second law of thermodynamics:

$$\int_{c(t_2)} s\, d^3x \geq \int_{c(t_1)} s\, d^3x, \qquad \forall t_2 > t_1$$

i.e., the entropy is a Liapunov function for the evolution c of an isolated system. We will, however, refer to the following postulate as the *Dynamical Postulate*:

Dynamical Postulate. The dynamics of an isolated system is supposed to be described by a vector field χ such that:

(a) $\chi = \chi^{\mathrm{H}} + \chi^{\mathrm{G}}$

with

$$\chi^{\mathrm{H}}(\hat{s}) = \nabla_i \zeta_s^{\mathrm{H}i}, \qquad \int_{\gamma} \chi^{\mathrm{G}}(\hat{s})\, d^3x \geq 0 \qquad \forall \gamma \in \mathscr{B}(\mathscr{E})$$

(b) The following relations hold:

$$\chi^{\mathrm{H}}(\hat{u}) = \nabla_i \zeta_u^{\mathrm{H}i}, \qquad \chi^{\mathrm{G}}(\hat{u}) = \nabla_i \zeta_u^{\mathrm{G}i} \tag{1}$$

$$\chi^{\mathrm{H}}(\hat{\pi}_i) = \nabla_j \zeta_{\pi i}^{\mathrm{H}j}, \qquad \chi^{\mathrm{G}}(\hat{\pi}_i) = \nabla_j \zeta_{\pi i}^{\mathrm{G}j} \tag{2}$$

$$\chi^{\mathrm{H}}(\hat{\rho}) = \nabla_i \zeta_\rho^{\mathrm{H}i}, \qquad \chi^{\mathrm{G}}(\hat{\rho}) = \nabla_i \zeta_\rho^{\mathrm{G}i} \tag{3}$$

$$\chi^{\mathrm{H}}(\hat{l}_i) = \nabla_m (\varepsilon_{ijk} \delta^{kl} x^j \zeta_{\pi l}^{Hm}), \qquad \chi^{\mathrm{G}}(\hat{l}_i) = \nabla_m (\varepsilon_{ijk} \delta^{kl} x^j \zeta_{\pi l}^{Gm}) \tag{4}$$

The conditions under (a) express that the dynamical vector field can be written as the sum of two terms, one that conserves the total entropy and one with respect to which the total entropy is a Liapunov function. The conditions under (b), moreover, indicate that each of the terms are vector fields for which the energy (1), the total momentum (2), the total mass (3), and the total angular momentum (4) are conserved.

4. THE HAMILTONIAN TERM χ^{H}

Let $\tilde{\mathscr{E}}$ be the fibered manifold

$$\tilde{\mathscr{E}} = \{(x^i, s_n, \Phi_n, u_{ni}, \phi^i_n) \varepsilon \cdots\} \to X = \{(x^i) \in \mathbb{R}^3\}$$

and let ω be the symplectic form on the fibers

$$\omega = \sum_{n=1}^{N} (d\Phi_n \wedge ds_n + d\phi^i_n \wedge du_{ni})$$

Moreover, denote by Ψ the submersion

$$J_1(\tilde{\mathscr{E}}) \to \mathscr{E}_u$$
$$(x^i, s_n, \Phi_n, u_{ni}, \phi^i_n)$$
$$\mapsto (x^i, s_n, \phi^j_{n,i} u_{nj} + s_n \Phi_{n,i}, j(\phi_n)|g|^{1/2} \phi^k_{n,i} \phi^l_{n,j} g_{kl}$$

where $j(\phi_n) = \det \phi^i_{n,j}$ and $g_{kl}(\phi_n)$ is a metric on $X = \mathbb{R}^3$.

The Hamiltonian of a system is by assumption the density

$$\hat{h} = \hat{u} \circ J_q(\Psi) : J_{q+1}(\tilde{\mathscr{E}}) \to \mathbb{R}$$

obtained by taking the pullback of its energy density $\hat{u} : J_q(\mathscr{E}_u) \to \mathbb{R}$.

Proposition. The pushforward under Ψ of the Hamiltonian vector field $(\nabla_{\Phi_n} \hat{h}, -\nabla_{s_n} \hat{h}, \nabla_{\phi^i_n} \hat{h}, -\nabla_{u_{ni}} \hat{h})$ is the vector field χ^{H} whose components are

$$\chi^{\mathrm{H}}(\hat{s}_n) = -\nabla_i(s_n \hat{v}^i_n)$$
$$\chi^{\mathrm{H}}(\hat{\pi}_{ni}) = -\nabla_j(\pi_{ni} \hat{v}^j_n) + 2\alpha_{nik} \hat{\sigma}^{kj}_n) - s_n \nabla_i \hat{T}_n - \pi_{nj} \nabla_i \hat{v}^j_n - \alpha_{njk} \nabla_i \hat{\sigma}^{jk}_n$$
$$\chi^{\mathrm{H}}(\hat{\alpha}_{nij}) = -\nabla_k(\alpha_{nij} \hat{v}^k_n) - \alpha_{njk} \nabla_i \hat{v}^k_n - \alpha_{nik} \nabla_j \hat{v}^k_n$$

Proof. The proposition is proved by computation using the formulas of paragraph 8 in Aaberge (1986). ■

Theorem. The vector field χ^{H} given above satisfies the conditions of the dynamical postulate.

Proof. Part (a) is automatically satisfied, since s is a constant of motion. Part (b) is proved by computation and inspection; thus [equation (1)]

$$\chi^{\mathrm{H}}(\hat{u}) = -\nabla_k \sum_{n=1}^{N} \{(\hat{T}_n s_n + \pi_{nj} \hat{v}^j_n + \alpha_{nij} \hat{\sigma}^{ij}_n) \hat{v}^k_n + 2\hat{v}^i_n \alpha_{nij} \hat{\sigma}^{jk}_n\}$$

and [equation (2)]

$$\chi^{\mathrm{H}}(\hat{\pi}_i) = -\nabla_j \left\{ \sum_{n=1}^{N} (\pi_{ni} \hat{v}^j_n + 2\alpha_{nik} \hat{\sigma}^{kj}_n + \hat{P}^j_{ni} \right.$$
$$\left. + \delta^j_i \left(-\hat{u} + \sum_{n=1}^{N} (s_n \hat{T}_n + \pi_{nj} \hat{v}^j_n + \alpha_{njk} \hat{\sigma}^{jk}_n \right) \right\}$$
$$= -\nabla_j \xi^{\mathrm{H}j}_i$$

where $P_n = P_{\pi_n}$ and the terms look like (Aaberge, 1986)

$$
\begin{aligned}
&p_i \partial p_j \hat{u} + \\
&p_{ii_1} \partial_{p_{i_1 j}} \hat{u} - p_i \nabla_{i_1} \partial_{p_{i_1 j}} \hat{u} + \\
&p_{ii_1 i_2} \partial_{p_{i_1 i_2 j}} \hat{u} - p_{ii_1} \nabla_{i_2} \partial_{p_{i_1 i_2 j}} \hat{u} + p_i \nabla_{i_1} \nabla_{i_2} \partial_{p_{i_1 i_2 j}} \hat{u} + \\
&\vdots \\
&p_{ii_1 i_2 \ldots i_{q-1}} \partial_{p_{i_1 i_2 \ldots i_{q-1} j}} \hat{u} - \cdots (-1)^{q-1} p_i \nabla_{i_1} \cdots \nabla_{i_{q-1}} \partial_{p_{i_1 \ldots i_{q-1} j}}
\end{aligned}
$$

For equation (3)

$$\chi^{\mathrm{H}}(\hat{\rho}) = -\nabla_i \left[\sum_{n=1}^{N} (\det \alpha_n)^{1/5} \hat{v}_n^i \right]$$

Equation (4) holds because $\hat{u}$ is invariant under rotations. It is thus a function of invariants only; accordingly,

$$\delta^{ik} \xi^{\mathrm{H}j}_{\pi k} = \delta^{jk} \xi^{\mathrm{H}i}_{\pi k} \quad \blacksquare$$

The vector field χ^{H} is a generalization of the Hamiltonian vector field computed in paragraph 4 of Aaberge (1986) (assuming no charge and electromagnetic field). In fact, let $\hat{u}$ depend on α_{nij} only through $\rho_n = (\operatorname{def} \alpha_{nij})^{1/5}$; then,

$$2\alpha_{nik} \hat{\sigma}_n^{kj} + \delta_i^j \alpha_{nkl} \hat{\sigma}_n^{kl} = \delta_i^j \hat{\rho}_n \nabla_{\rho_n} \hat{u}$$

This shows also that χ^{H} is a generalization of the Euler vector field of a one-component fluid.

5. THE EQUILIBRIUM CONDITIONS

The vector field χ^{H} leaves invariant the entropy and thus does not alone satisfy the second law. In fact, no Hamiltonian vector field could, since Hamiltonian vector fields do not possess attractors. We must therefore add to χ^{H} a vector field χ^{G} that is also tangent to the equienergy, equimomentum, equimass, and equi-angular momentum submanifolds, but which has the maxima of the entropy on these submanifolds as attractors. In fact, these maxima are critical points of χ^{H}.

To construct χ^{G} we will first consider the conditions determining the extrema of the entropy on the submanifolds of given energy, momentum, mass, and angular momentum. These are partly determined by the zeros of the first derivatives (variations) of the entropy with respect to vector fields tangent to the above submanifolds.

The class τ of variations to consider consists of (1) the local deformations generated by the vector fields of the form (Aaberge, 1986)

$$(\nabla_i(u\xi^i), \nabla_i \xi^j \pi_{nj} + \nabla_j(\pi_{ni}\xi^j), \nabla_i(\sigma_n \xi^i), \nabla_k(\alpha_{nij}\xi^k) + \nabla_i \xi^k \alpha_{nkj} + \nabla_j \xi^k \alpha_{nki})$$

where ξ^i is a vector field on X tangent to the boundary of the domain containing the system; and (2) of the vector fields χ with components of the form

$$\chi(\hat{u}) = \nabla_i \zeta^i_u, \qquad \chi(\hat{\pi}_i) = \nabla_j \zeta^j_{\pi\, i} + \delta^{jl}\delta_{ik}\nabla_j \zeta^k_{\pi\, l}$$

$$\chi(\hat{p}_{mi}) = \chi_{p_{mi}}, \qquad \chi(\hat{\sigma}_m) = \chi_{\sigma_m}, \qquad \chi(\hat{\rho}) = \nabla_i \zeta^i_\rho, \qquad \chi(\hat{\eta}_m) = \chi_{\eta_m}$$

$$\chi(\hat{\beta}_{nij}) = \tfrac{2}{3}\beta_{nij}\nabla_k \zeta^k_\beta - \beta_{nkj}\nabla_i \zeta^k_\beta - \beta_{nki}\nabla_j \zeta^k_\beta$$

where we have introduced a new set of coordinates defined by

$$\pi_i = \sum_{n=1}^{N} \pi_{ni}$$

$$p_{mi} = \frac{\sum_{\gamma=1}^{m} \rho_\gamma \pi_{m+1\, i} - \rho_{m+1} \sum_{\gamma=1}^{m} \pi_{\gamma i}}{\sum_{\gamma=1}^{m} \rho_\gamma}, \qquad m = 1, \ldots, N-1$$

$$\rho = \sum_{n=1}^{N} (\det \alpha_n)^{1/5}$$

$$\eta_m = \sum_{\gamma=1}^{m} (\det \alpha_n)^{1/5}, \qquad m = 1, \ldots, N-1$$

$$\beta_{nij} = \det(\alpha_n)^{-1/3}\alpha_{nij}$$

The extremal condition $\chi(s) = 0$, $\forall \chi \in \tau$, then read as follow.

1. The variation by local deformations give

$$\nabla_k\left(\frac{1}{T}\xi^{\mathrm{H}k}_i\right) = 0$$

2. The variations in the other directions give

$$\nabla_i \partial_u \hat{s} = 0$$

$$\nabla_i \delta_{kj} \nabla_{\pi_k} \hat{s} + \nabla_j \delta_{ki} \nabla_{\pi_k} \hat{s} = 0$$

$$\nabla_{p_{mi}} \hat{s} = 0$$

$$\partial_{\sigma_n} \hat{s} = 0$$

$$\nabla_i \nabla_\rho \hat{s} = 0$$

$$\nabla_{\eta_m} \hat{s} = 0$$

$$\nabla_k(\beta_{nij}\nabla_{\beta_{nkj}}\hat{s} - \tfrac{1}{3}\delta^k_i \beta_{njl}\nabla_{\beta_{njl}}\hat{s}) = 0$$

We notice that when s depends on β_{nij} only through $\det \beta_{nij}$ $(=1)$, then

$$\beta_{nij}\nabla_{\beta_{nkj}}\hat{s} - \tfrac{1}{3}\delta^k_i \beta_{nkl}\nabla_{\beta_{nkl}}\hat{s} = 0$$

and the extremal conditions above reduce to the extremal conditions for an N-component fluid (paragraph Aaberge, 1986).

6. THE CONSTRUCTION OF THE GRADIENT TERM χ^G IN THE DYNAMICAL VECTOR FIELD

Let $\mathscr{F}$ denote the source space for the intensive variables of the entropy representation and of the variables β_{nij},

$$\mathscr{F}=\{(\beta, \Omega^i, \omega^i_n, \tau_m, \nu, \nu_m, \gamma^{ij}_n, \beta_{nij})\varepsilon \dots\}$$

and let ϕ_s denote the map

$$\begin{aligned}&J_q(\mathscr{E}_s)\to\mathscr{F}\\ &(u, \pi_i, \dots, \beta_{nij,i_1 i_2\dots i_{q-1}})\\ &\mapsto(\partial_u\hat{s}, -\nabla_{\pi_i}\hat{s}, \dots, -\nabla_{\beta_{nij}}\hat{s}, \beta_{nij})\end{aligned}$$

The "global" extremal conditions can then be written

$$\hat{\beta}_{,i}\circ J_1\phi_s=0$$

$$(\delta_{kj}\hat{\Omega}^k_{,i}+\delta_{ki}\hat{\Omega}^k_{,j})\circ J_1\phi_s=0$$

$$\hat{\omega}^i\circ\phi_s=0,\qquad \hat{\tau}_m\circ\phi_s\times 0,\qquad \hat{\nu}_{,i}\circ J_1\phi_s=0,\qquad \nu_m\circ\phi_s=0$$

$$(\beta_{nij}\hat{\gamma}^{kj}_n-\tfrac{1}{3}\delta^k_i\beta_{njl}\tilde{\gamma}^{jl}_n)_{,k}\circ J_1\phi_s=0$$

Let $\psi: J_1\mathscr{F}\to\mathscr{M}$ denote the submersion defined by

$$\begin{aligned}\psi^1&=\tfrac{1}{2}\delta^{ij}\beta^{-3}\beta_{,i}\beta_{,j}\\ \psi^2&=\tfrac{1}{4}\beta\delta^{ki}\delta^{lj}\{(\beta^{-1}\delta_{pj}\Omega^p)_{,i}+(\beta^{-1}\delta_{pi}\Omega^p)_{,j}\}\\ &\quad\times\{(\beta^{-1}\delta_{qk}\Omega^q)_{,l}+(\beta^{-1}\delta_{ql}\Omega^q)_{,k}\}\\ \psi^3&=\tfrac{1}{2}\beta\{(\beta^{-1}\Omega^i)_{,i}\}^2\\ \psi^4_m&=\tfrac{1}{2}\beta\delta_{ij}\omega^i_m\omega^j_m\\ \psi^5_m&=\tfrac{1}{2}\beta\tau^2_m\\ \psi^6&=\tfrac{1}{2}\beta\delta^{ij}(\beta^{-1}\nu_{,i}(\beta^{-1}\nu)_{,j}\\ \psi^7_m&=\tfrac{1}{2}\beta\nu^2_m\\ \psi^8_n&=\tfrac{1}{2}\beta\delta^{ij}\{\beta^{-1}(\beta_{nil}\gamma^{kl}_n-\tfrac{1}{3}\delta^k_i\beta_{nlo}\gamma^{lo}_n)\}_{,k}\\ &\quad\times\{\beta^{-1}(\beta_{njp}\gamma^{qp}_n-\tfrac{1}{3}\delta q_j\beta_{npr}\gamma^{pr}_n)\}_{,q}\end{aligned}$$

and let $r:\mathscr{M}\to\mathbb{R}$. By pullback with ψ we obtain a function

$$\tilde{r}=r\circ\psi: J_1\mathscr{F}\to\mathbb{R}$$

We define a family of functions by dividing the variables of $\tilde{r}$ into two groups, i.e.,

$$\tilde{r}_{(\beta,\Omega^i,\nu,\gamma^{ij}_n,\beta_{nij},\beta_{nij,k})}\beta_{,i}, \Omega^i_{,j}, \omega^i, \tau_m, \nu_{,i}, \nu_m, \gamma^{ij}_{n,k})$$

The gradient of $\tilde{r}_{()}$ is by definition

$$\operatorname{grad} \tilde{r}_{()} = (\nabla_\beta \tilde{r}_{()}, -\nabla_{\Omega^i} \tilde{r}_{()}, -\nabla_{\omega_m^i} \tilde{r}_{()}, -\nabla_{\tau_m} \tilde{r}_{()},$$
$$-\nabla_\nu \tilde{r}_{()}, -\nabla_{\nu_m} \tilde{r}_{()}, -\nabla_{\gamma_n^{ij}} \tilde{r}_{()})$$

and χ^{G} is assumed to be given by

$$\chi^{\mathrm{G}} = \operatorname{grad} \tilde{r} \circ J_2 \phi_s$$

Thus, computing the components of χ^{G}, we get

$$\chi^{\mathrm{G}}(\hat{u}) = \nabla_i \{\delta^{ij} \kappa \nabla_j T + \delta_{jk} \Sigma^{ij} V^k + \mu \partial_{\psi^6} r \mu_{,j}$$
$$+ \sum_n \delta_{kl} \partial_{\psi_n^8} r c_n^{ki} c_{n,j}^{lj}\}$$

$$\chi^{\mathrm{G}}(\pi_i) = \delta_{ki} \nabla_j \Sigma^{kj}$$

$$\chi^{\mathrm{G}}(p_{mi}) = -\partial_{\psi_m^4} r \delta_{ij} v_m^j$$

$$\chi^{\mathrm{G}}(\sigma_m) = -\partial_{\psi_m^5} r t_m$$

$$\chi^{\mathrm{G}}(\rho) = \nabla_i (\partial_{\psi^6} r \delta^{ij} \mu_{,j})$$

$$\chi^{\mathrm{G}}(\eta_m) = -\partial_{\psi_m^7} r \mu_m$$

$$\chi^{\mathrm{G}}(\beta_{nij}) = -\tfrac{2}{3} \beta_{nij} \nabla_k (\partial_{\psi_n^8} r c_{n,l}^{kl}) + \beta_{nki} \nabla_j (\partial_{\psi_n^8} r c_{n,l}^{kl})$$

where

$$\Sigma^{ij} = \eta(\delta^{ik} V^j_{,k} + \delta^{jk} V^i_{,k} - \tfrac{2}{3} \delta^{ij} V^k_{,k}) + \xi \delta^{ij} V^k_{,k}$$

$$\eta = \partial_{\psi^2} r$$

$$\xi = \tfrac{2}{3} \partial_{\psi^2} r + \partial_{\psi^3} r$$

$$\kappa = \partial_{\psi^1} r$$

$$c_n^{ij} = \delta^{ik} (\beta_{nkl} \sigma_n^{lj} - \tfrac{1}{3} \delta^j_k \beta_{nkl} \sigma_n^{kl})$$

when we express the results in terms of the intensive variables of the energy representation corresponding to the following choice:

$$(s, \pi_i, p_{mi}, \sigma_m, \rho, \eta_m, \beta_{nij})$$

of extensive variables. Thus, T denotes the global temperature, V^i the velocity of the center of mass, μ the global chemical potential, v_m^i relative velocities, t_m relative temperatures, and μ_n relative chemical potentials.

Using the above expressions for the components of χ^{G}, we can compute $\chi^{G}(\hat{s})$. The result is (see paragraph 49 in Landau and Lifschitz, 1971)

$$\chi^{G}(\hat{s}) = \nabla_i(\delta^{ij}\kappa T_{,j}) + \frac{1}{T}\kappa\delta^{ij}T_{,i}T_{,j}$$
$$+\frac{1}{2}\frac{1}{T}\eta\delta^{ki}\delta^{lj}\left(\delta_{pj}V^{p}{}_{,i} + \delta_{pi}V^{p}{}_{,j} - \frac{2}{3}\delta_{ij}V^{p}{}_{,p}\right)$$
$$\times\left(\delta_{qk}V^{q}_{,l} + \delta_{ql}V^{q}_{,k} - \frac{2}{3}\delta_{kl}V^{q}_{,q}\right)$$
$$+\frac{1}{T}\xi(V^{i}_{,i})^{2} + \frac{1}{T}\sum_{m}\partial_{\psi^{4}_{m}}r\delta_{ij}v^{i}_{m}v^{j}_{m} + \frac{1}{T}\sum_{m}\partial_{\psi^{7}_{m}}r\mu^{2}_{m}$$
$$+\frac{1}{T}\sum_{m}\partial_{\psi^{5}_{m}}rt^{2}_{m} + \frac{1}{T}\partial_{\psi^{6}}r\delta^{ij}\mu_{,i}\mu_{,j} + \frac{1}{T}\sum_{n}\partial_{\psi^{8}_{n}}r\delta_{ij}c^{ik}_{n,k}c^{jl}_{n,l}$$

i.e., $\chi^{G}(s)$ is the sum of a convection term and quadratic terms that can be regrouped such that their coefficients are $(1/T)\partial_{\psi}r$.

Theorem. A vector field χ^{G} constructred according to the above prescriptions satisfies the dynamical postulate if $\partial_{\psi}{}^{\alpha}r > 0\ \forall\alpha$.

Proof. By inspection. We also notice that $\chi^{G} \equiv 0$ on the equilibrium states. ■

ACKNOWLEDGMENTS

I am indebted to C. Piron for numerous discussions on the subject of this paper. I also wish to thank the members of the Département de Physique for warm hospitality, and the Fonds National Suisse for support.

REFERENCES

Aaberge, T. (1986). On the equations of motion of an N-component charged fluid, *Helvetica Physica Acta*, **59**, 390–409 (1986).

Holm, D. D., and Kuperschmidt, B. A. (1983). Poisson brackets and Clebsch representations for magnetohydrodynamics, multifluid plasmas, and elasticity, *Physica* **6D**, 374–363.

Palais, R. S. (1968). *Foundations of Global Non-linear Analysis*, Benjamin, New York.

Pommaret, J. F. (1978). *Systems of Partial Differential Equations and Lie Pseudogroups*, Gordon and Breach, New York.

Landau, L., and Lifschitz, E. (1971). *Mécanique des Fluids*, Mir, Moscow.